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Connecting orbits between static classes for generic Lagrangian systems

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Abstract

Let L be a C^∞ convex superlinear Lagrangian on a closed manifold M. We show that if the number of static classes is finite, then there exist chains of semistatic orbits that connect any two given static classes. Using this property we show that if there is only one static class, then the homoclinic orbits to the set of static orbits generate over $\mathbb{R}$ the relative homology of the pair (M, U), where U is a sufficiently small connected neighborhood of the set of static orbits in M. We show that generically in the sense of Mañé (in: F. Ledrappier, J. Lewowicz, S. Newhouse (Eds.), International Congress on Dynamical Systems in Montevideo (a tribute to Ricardo Mañé), Pitman Research Notes in Mathematics, Vol. 362, 1996, pp. 120–131 (reprinted in Bol. Soc. Bras. Mat. 28(2) (1997) 141–157) the set of semistatic orbits coincides with the support of a uniquely minimizing measure, therefore generically, the homoclinic orbits to the support of the minimizing measure generate over $\mathbb{R}$ the relative homology of the pair (M, U), where U is a sufficiently small connected neighborhood of the projection of the support of the measure to M. This last result was obtained—with a different proof—by Bolotin (Proceedings of the International Congress of Mathematics, Vol. 1,2, Zürich, 1994, Birkhäuser, Basel, 1995, pp. 1169–1178; in: V.V. Kozlov (Ed.), Dynamical Systems in Classical Mechanics, American Mathematical Society Translation Series 2, Vol. 168, American Mathematical Society, Providence, RI, 1995, pp. 21–90) assuming the existence of a $C^{1+\mathrm{Lip}}$ function $f: M \to \mathbb{R}$ such that $L + c - df \geqslant 0$, where c is the critical value of L. Finally, we obtain two consequences. The first one says that if M is a closed manifold with first Betti number $\geqslant 2$ then there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the Lagrangian $L + \psi$ has a unique minimizing measure and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and the stable and unstable manifolds have transverse homoclinic intersections. The second consequence says that if M is a closed manifold with first Betti number different from zero and if L is a symmetric Lagrangian, then there exists a generic set

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$\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$, then $L + \psi$ has a unique minimizing measure and this measure is supported on a hyperbolic fixed point whose stable and unstable manifolds have transverse homoclinic intersections.

MSC: 58F05; 58F11

1. Introduction

Let M be a closed connected smooth manifold and let $L: TM \to \mathbb{R}$ be a smooth convex superlinear Lagrangian. This means that L restricted to each T_xM has positive definite Hessian and that for some Riemannian metric we have that

$$\lim_{|v| \to \infty} \frac{L(x, v)}{|v|} = \infty,$$

uniformly on $x \in M$. Since M is compact, the extremals of L give rise to a complete flow $f_t: TM \to TM$ called the Euler–Lagrange flow of the Lagrangian. The extremals are solutions of the Euler–Lagrange equation which in local coordinates is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial v} = \frac{\partial L}{\partial x}. \tag{E–L}$$

The *energy* $E: TM \to \mathbb{R}$ is defined by

$$E(x, v) = \frac{\partial L}{\partial v}(x, v)v - L(x, v).$$

Since L is autonomous, E is a first integral of the flow f_t.

Recall that the action of the Lagrangian L on an absolutely continuous curve $\gamma: [a, b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t.$$

Given two points, x_1 and x_2 in M and $T > 0$ denote by $\mathscr{C}_T(x_1, x_2)$ the set of absolutely continuous curves $\gamma: [0, T] \to M$, with $\gamma(0) = x_1$ and $\gamma(T) = x_2$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k: M \times M \to \mathbb{R}$ by

$$\Phi_k(x_1, x_2) = \inf\left\{A_{L+k}(\gamma):\ \gamma \in \bigcup_{T>0} \mathscr{C}_T(x_1, x_2)\right\}.$$

The *critical value of* L, which was introduced by Mañé [10], is the real number $c(L)$ defined as the infimum of $k \in \mathbb{R}$ such that for some $x \in M$, $\Phi_k(x, x) > -\infty$. Since L is convex and superlinear and M is compact such a number exists and it has various important properties that we review in

Section 2. We briefly mention a few of them since we shall need them below. For any $k \geqslant c(L)$, the action potential Φ_k is a Lipschitz function that satisfies a triangle inequality. In general, the action potential is not symmetric but if we define $d_k : M \times M \to \mathbb{R}$ by setting

$$d_k(x, y) = \Phi_k(x, y) + \Phi_k(y, x),$$

then d_k is a distance function for all $k > c(L)$ and a pseudo-distance for $k = c(L)$.

Since $d_k \geqslant 0$, for every absolutely continuous curve $\gamma : [a, b] \to M$ and all $k \geqslant c(L)$ we have

$$A_{L+k}(\gamma) \geqslant \Phi_k(\gamma(a), \gamma(b)) \geqslant -\Phi_k(\gamma(b), \gamma(a)). \tag{1}$$

Set $c = c(L)$. We say that an absolutely continuous curve $\gamma : [a, b] \to M$ is *semistatic* if

$$A_{L+c}(\gamma|_{[t_0,t_1]}) = \Phi_c(\gamma(t_0), \gamma(t_1))$$

for all $a < t_0 \leqslant t_1 < b$; and that it is *static* if

$$A_{L+c}(\gamma|_{[t_0,t_1]}) = -\Phi_c(\gamma(t_1), \gamma(t_0))$$

for all $a < t_0 \leqslant t_1 < b$. Clearly, by (1) a static curve is semistatic. One could also say that $\gamma|_{[a,b]}$ is static if it is semistatic and for all $a < t_0 \leqslant t_1 < b$, $d_c(\gamma(t_0), \gamma(t_1)) = 0$. Semistatic curves are solutions of the Euler–Lagrange equation because of their minimizing properties. Also it is not hard to check that semistatic curves have energy precisely c [10,4]. The notions of semistatic and static curves are closely related to Mather's notions of *c-minimal trajectories* and *regular c-minimal trajectories*, respectively (see [13]).

Given a vector $v \in TM$ we shall denote by $x_v : \mathbb{R} \to M$ the solution of the Euler–Lagrange equation with $\dot{x}(0) = v$.

The set of vectors v in TM that give rise to static curves $x_v : \mathbb{R} \to M$ is a closed invariant set that we shall denote by $\hat{\Sigma} := \hat{\Sigma}(L)$. Similarly, the set of vectors v in TM that give rise to semistatic curves $x_v : \mathbb{R} \to M$ is a closed invariant set that we shall denote by $\Sigma := \Sigma(L)$. The set $\hat{\Sigma}$ is chain recurrent and the set Σ is chain transitive [10,4, Theorem V]. As we mentioned before $\hat{\Sigma} \subset \Sigma$. We need to recall (cf. Section 3, Theorem 3.2) the following important *Lipschitz graph property* which was shown in [10,4] and [13, Theorem 6.1] that generalizes the celebrated Lipschitz Graph Theorem of Mather [12]: the set $\hat{\Sigma}$ is a Lipschitz graph, that is, if $\pi : TM \to M$ denotes the canonical projection then the map $\pi|_{\hat{\Sigma}} : \hat{\Sigma} \to \pi(\hat{\Sigma})$ is bijective with Lipschitz inverse. Using the graph property we can define an equivalence relation in $\hat{\Sigma}$ by saying that two vectors v and w in $\hat{\Sigma}$ are equivalent iff $d_c(\pi(v), \pi(w)) = 0$. The equivalence relation breaks $\hat{\Sigma}$ into classes that we shall call *static classes*. Let Λ be the set of static classes. Define a reflexive partial order $\preccurlyeq$ in Λ by

(a) $\preccurlyeq$ is reflexive.
(b) $\preccurlyeq$ is transitive.
(c) If there is $v \in \Sigma$ with the α-limit set $\alpha(v) \subseteq \Lambda_i$ and ω-limit set $\omega(v) \subseteq \Lambda_j$, then $\Lambda_i \preccurlyeq \Lambda_j$.

Theorem A. *Suppose that the number of static classes is finite. Then given Λ_i and Λ_j in Λ, we have that $\Lambda_i \preccurlyeq \Lambda_j$.*

Theorem A could be restated by saying that if the cardinality of Λ is finite, then given two static classes Λ_i and Λ_j there exist classes $\Lambda_i = \Lambda_1, \ldots, \Lambda_n = \Lambda_j$ and semistatic vectors $v_1, \ldots, v_{n-1} \in \Sigma$

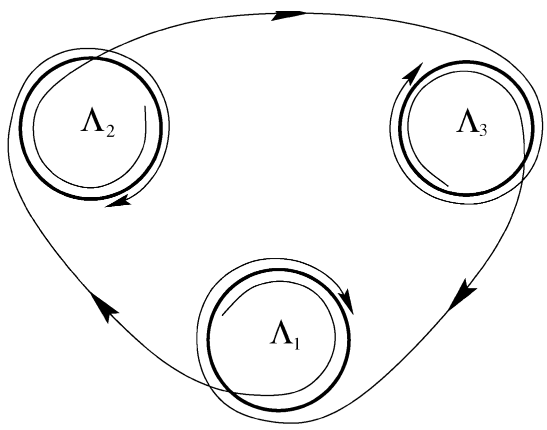

Fig. 1. Connecting orbits between static classes. The three closed curves represent the static classes and the other curves represent semistatic orbits connecting them.

such that for all $1 \leqslant k \leqslant n-1$ we have that $\alpha(v_k) \subseteq \Lambda_k$ and $\omega(v_k) \subseteq \Lambda_{k+1}$. In other words, between two static classes there exists a chain of static classes connected by heteroclinic semistatic orbits (cf. Fig. 1).

Let us assume now that $\hat{\Sigma}$ contains only *one* static class. We shall see in Section 3 (cf. Proposition 3.4) that the static classes are always connected, thus if we assume that there is only one static class, $\hat{\Sigma}$ must be connected.

Given $\varepsilon > 0$, let U_ε be the ε-neighbourhood of $\pi(\hat{\Sigma})$. Since $\hat{\Sigma}$ is connected, the open set U_ε is connected for ε sufficiently small. Let $H_1(M, U_\varepsilon, \mathbb{R})$ denote the first relative singular homology group of the pair (M, U_ε) with real coefficients.

We shall say that an orbit of L is *homoclinic* to a closed invariant set $K \subset TM$ if its α and ω-limit sets are contained in K.

Observe that to each homoclinic orbit $x : \mathbb{R} \to M$ to the set of static orbits $\hat{\Sigma}$ we can associate a homology class in $H_1(M, U_\varepsilon, \mathbb{R})$. Indeed, since there exists $t_0 > 0$ such that for all t with $|t| \geqslant t_0$, $x(t) \in U_\varepsilon$, the class of $x|_{[-t_0, t_0]}$ defines an element in $H_1(M, U_\varepsilon, \mathbb{R})$. Let us denote by $\mathscr{H}$ the subset of $H_1(M, U_\varepsilon, \mathbb{R})$ given by all the classes corresponding to homoclinic orbits to $\hat{\Sigma}$.

In Section 4, we shall show the following result.

Theorem B. *Suppose that $\hat{\Sigma}$ contains only one static class. Then for any ε sufficiently small the set $\mathscr{H}$ generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least* $\dim H_1(M, U_\varepsilon, \mathbb{R})$ *homoclinic orbits to the set of static orbits $\hat{\Sigma}$.*

In [11], Mañé introduced the concept of *generic property of a Lagrangian*. A property P is said to be *generic* for the Lagrangian L if there exists a generic set $\mathcal{O}$ (in the Baire sense) of the set $C^\infty(M, \mathbb{R})$ of all C^∞ functions from M to $\mathbb{R}$ such that if $\psi \in \mathcal{O}$ the Lagrangian $L + \psi$ has the property P. One of Mañé's objectives in [11] was to show that Mather's theory of minimizing measures becomes much more accurate and stronger if one searches for generic properties.

Our next result describes a generic property of Lagrangians on closed manifolds. Let $\mathscr{M}(L)$ be the set of probabilities on the Borel σ-algebra of TM that have compact support and are invariant under the flow f_t. We shall say that a measure $\mu \in \mathcal{M}(L)$ is *minimizing* if

$$\int L \, \mathrm{d}\mu = -c.$$

We shall denote by $\mathcal{M}^0(L)$ the set of minimizing measures. We say that a measure μ is *uniquely minimizing* if the set $\mathcal{M}^0(L)$ contains μ only. It was shown in [10,4] that a measure μ is minimizing if and only if the support of μ is contained in $\hat{\Sigma}$. Mather has shown in [13] that if μ is a minimizing measure then its support is contained in the set of Mather's regular c-minimal curves.

The following important generic property was proved in [5,11]. Given a Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M,\mathbb{R})$ such that if $\psi \in \mathcal{O}$ the Lagrangian $L + \psi$ has a unique minimizing measure in $\mathcal{M}^0(L + \psi)$ and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and *if* the stable and unstable manifolds intersect, they must do it transversally. It is conjectured in [10] that the unique minimizing measure in $\mathcal{M}^0(L + \psi)$ is always supported on a periodic orbit.

We will prove in Section 5:

Theorem C. *Let*

$$\mathcal{G}_2 := \{\psi \in C^\infty(M, \mathbb{R}) \mid \mathcal{M}^0(L + \psi) = \{\mu\} \text{ and } \operatorname{supp}(\mu) = \hat{\Sigma}(L + \psi) = \Sigma(L + \psi)\}.$$

Then,

(a) *$\mathcal{G}_2$ is generic in $C^\infty(M, \mathbb{R})$.*

(b) *If $\psi_0 \in \mathcal{G}_2$, then $\lim_{\psi \to \psi_0} d_{\mathrm{H}}(\hat{\Sigma}(L + \psi), \hat{\Sigma}(L + \psi_0)) = 0$, where d_{H} is the Hausdorff metric between compact subsets of TM.*

(c) *If $\psi \in C^\infty(M, \mathbb{R})$, $\mu_\psi \in \mathcal{M}^0(L + \psi)$ and $\psi_0 \in \mathcal{G}_2$, then*

$$\lim_{\psi \to \psi_0} d_{\mathrm{H}}(\operatorname{supp}(\mu_\psi), \operatorname{supp}(\mu_{\psi_0})) = 0.$$

Note that since μ is also uniquely ergodic, the set $\operatorname{supp}(\mu)$ must be a static class. Therefore, generically, the set of static orbits contains only one static class and it coincides with the support of the uniquely minimizing measure.

Let us denote by U_ε the ε-neighborhood of the set $\operatorname{supp}(\mu)$. From Theorems B and C we obtain right away the following generic property.

Corollary 1. *Given a Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the Lagrangian $L + \psi$ has a unique minimizing measure μ in $\mathcal{M}^0(L + \psi)$ and this measure is uniquely ergodic. For any ε sufficiently small the set $\mathcal{H}$ of homoclinic orbits to* $\operatorname{supp}(\mu)$ *generates over $\mathbb{R}$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, there exist at least* $\dim H_1(M, U_\varepsilon, \mathbb{R})$ *homoclinic orbits to* $\operatorname{supp}(\mu)$.

Bolotin has shown in [1, Theorem 3.4] and [2, Theorem 4.3] (cf. also [3]) that if there exists a $C^{1+\mathrm{Lip}}$ function $f: M \to \mathbb{R}$ such that

$$L + c - df \geqslant 0,$$

then the set $\mathscr{H}$ of homoclinic orbits to supp(μ) generates over $\mathbb{R}^+$ the relative homology $H_1(M, U_\varepsilon, \mathbb{R})$. In particular, he gets at least $2 \dim H_1(M, U_\varepsilon, \mathbb{R})$ homoclinic orbits to supp(μ), twice as much as we do in Corollary 1. However, we do not know if his condition is generic.

Bolotin uses methods different from ours. To prove Theorem B we consider finite coverings M_0 of M whose group of deck transformations is given by the quotient of $H_1(M, U_\varepsilon, \mathbb{Z})$/(torsion) by a finite index subgroup. Using that the lifted Lagrangian L_0 has the same critical value as L, we conclude that the number of static classes of L_0 must be finite. Hence we can apply Theorem A to L_0 to deduce that the group generated by the homoclinic orbits to the set of static orbits of L coincides with $H_1(M, U_\varepsilon, \mathbb{R})$.

We note that the homoclinic orbits that we obtain in Theorem B and Corollary 1 have energy c but they are not semistatic orbits of L (cf. Theorem C). However, they are semistatic for lifts of L to suitable finite covers.

Using Corollary 1, we shall show in Section 6:

Corollary 2. *Let M be a closed manifold with first Betti number $\geqslant 2$. Given a Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$ the Lagrangian $L + \psi$ has a unique minimizing measure in $\mathscr{M}^0(L+\psi)$ and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and the stable and unstable manifolds have transverse homoclinic intersections.*

We say that a Lagrangian L is *symmetric* if for all $(x, v) \in TM$, $L(x, v) = L(x, -v)$. Note that if L is symmetric and $\psi \in C^\infty(M, \mathbb{R})$ then $L + \psi$ is also symmetric.

Corollary 3. *Let M be a closed manifold with first Betti number different from zero. Given a symmetric Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M, \mathbb{R})$ such that if $\psi \in \mathcal{O}$, then $L + \psi$ has a unique minimizing measure in $\mathscr{M}^0(L+\psi)$ and this measure is supported on a hyperbolic fixed point whose stable and unstable manifolds have transverse homoclinic intersections.*

In [8] Albert Fathi has obtained independently results which have a considerable overlap with Theorem B. He defines a set $\mathscr{C}_0$ by

$$\mathscr{C}_0 = \bigcup_p \mathrm{d}p(\Sigma_0),$$

where the union is taken over all finite and abelian Galois covers $p: M_0 \to M$ and Σ_0 is the set of semistatic orbits of the lift of L to M_0. He shows that the connected invariant set $\mathscr{C}_0$ is contained in $W^s(\hat{\Sigma}) \cap W^u(\hat{\Sigma})$ and that for any connected open set V containing $\mathscr{C}_0$ one has $H_1(TM, V, \mathbb{Z}) = 0$. As a corollary, he also obtains the existence of at least $\dim H_1(M, U_\varepsilon, \mathbb{R})$ homoclinic orbits to the set of static orbits $\hat{\Sigma}$ and without assuming that $\hat{\Sigma}$ contains only one static class.

At this point, it seems useful to note that there are various terminologies in the literature for several of the concepts that we use here. Fathi refers in [6–9] to the closure of the union of the support of minimizing measures as the *Aubry–Mather set*. What we call here the static and semistatic sets, Fathi calls the *Peierls set* and the *Mañé set*, respectively. As we mentioned before, semistatic and static curves are closely related to Mather's notions of *c-minimal trajectories* and *regular c-minimal trajectories*, respectively. The terminology we follow in this paper is that of Mañé in [10].

2. Critical values, static and semistatic curves

Let M be a closed connected manifold and $L: TM \to \mathbb{R}$ a convex superlinear Lagrangian.

The action of the Lagrangian L on an absolutely continuous curve $\gamma:[a,b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t.$$

Given two points, x and y in M and $T > 0$ denote by $\mathscr{C}_T(x,y)$ the set of absolutely continuous curves $\gamma:[0,T] \to M$, with $\gamma(0) = x$ and $\gamma(T) = y$. For each $k \in \mathbb{R}$ we define the *action potential* $\Phi_k : M \times M \to \mathbb{R}$ by

$$\Phi_k(x,y) = \inf\left\{ A_{L+k}(\gamma) : \gamma \in \bigcup_{T>0} \mathscr{C}_T(x,y) \right\}.$$

Theorem 2.1 (Basic properties of the critical value; Contreras et al. [4]; Mañé [10]). *There exists* $c(L) \in \mathbb{R}$ *such that*

1. *if* $k < c(L)$, *then* $\Phi_k(x_1, x_2) = -\infty$, *for all* x_1 *and* x_2 *in* M;
2. *if* $k \geqslant c(L)$, *then* $\Phi_k(x_1, x_2) > -\infty$ *for all* x_1 *and* x_2 *in* M *and* Φ_k *is a Lipschitz function*;
3. *if* $k \geqslant c(L)$, *then*
$$\Phi_k(x_1, x_3) \leqslant \Phi_k(x_1, x_2) + \Phi_k(x_2, x_3)$$
for all x_1, x_2 *and* x_3 *in* M *and*
$$\Phi_k(x_1, x_2) + \Phi_k(x_2, x_1) \geqslant 0,$$
for all x_1 *and* x_2 *in* M;
4. *if* $k > c(L)$, *then for* $x_1 \neq x_2$ *we have*
$$\Phi_k(x_1, x_2) + \Phi_k(x_2, x_1) > 0.$$

Observe that in general the action potential Φ_k is *not* symmetric; however, defining $d_k : M \times M \to \mathbb{R}$ by

$$d_k(x,y) = \Phi_k(x,y) + \Phi_k(y,x),$$

Theorem 2.1 says that d_k is a metric for $k > c(L)$ and a pseudometric for $k = c(L)$. The number $c(L)$ is called the *critical value* of L.

It is important for our purposes to indicate that the theorem above also holds for coverings of M, i.e. suppose $\hat{M}$ is a covering of M with covering projection p. Take the lift of the Lagrangian L to $\hat{M}$ which is given by

$$\hat{L}(x,v) = L(p(x), \mathrm{d}p(v)).$$

Then we define for each $k \in \mathbb{R}$ the action potential just as above and the results hold for $\hat{L}$. Thus, we have a critical value for $\hat{L}$.

Using the theorem it is straightforward to check that if M_1 and M_2 are coverings of M such that M_1 covers M_2, then

$$c(L_1) \leqslant c(L_2), \tag{2}$$

where L_1 and L_2 denote the lifts of the Lagrangian L to M_1 and M_2, respectively. Also we have the following lemma.

Lemma 2.2. *If M_1 is a finite covering of M_2 then $c(L_1) = c(L_2)$.*

Proof. We know that $c(L_1) \leqslant c(L_2)$. Suppose that the strict inequality holds and let k be such that $c(L_1) < k < c(L_2)$. Hence, there exists a closed curve γ in M_2 with negative $(L_2 + k)$-action. Since M_1 is a finite covering of M_2 some iterate of γ lifts to a closed curve in M_1 with negative $(L_1 + k)$-action which contradicts $c(L_1) < k$. $\square$

Note that for every absolutely continuous curve $\gamma:[a,b] \to M$ and all $k \geqslant c(L)$ Theorem 2.1 implies that

$$A_{L+k}(\gamma) \geqslant \Phi_k(\gamma(a),\gamma(b)) \geqslant -\Phi_k(\gamma(b), \gamma(a)). \tag{3}$$

Set $c = c(L)$. We say that an absolutely continuous curve $\gamma:[a,b] \to M$ is *semistatic* if

$$A_{L+c}(\gamma|_{[t_0,t_1]}) = \Phi_c(\gamma(t_0), \gamma(t_1))$$

for all $a < t_0 \leqslant t_1 < b$; and that is *static* if

$$A_{L+c}(\gamma|_{[t_0,t_1]}) = -\Phi_c(\gamma(t_1), \gamma(t_0))$$

for all $a < t_0 \leqslant t_1 < b$. Clearly, by (3) a static curve is semistatic. One could also say that $\gamma|_{[a,b]}$ is static if it is semistatic and for all $a < t_0 \leqslant t_1 < b$, $d_c(\gamma(t_0), \gamma(t_1)) = 0$. Semistatic curves are solutions of the Euler–Lagrange equation because of their minimizing properties. Also it is not hard to check that semistatic curves have energy precisely c [10,4].

Given a vector $v \in TM$ we shall denote by $x_v : \mathbb{R} \to M$ the solution of the Euler–Lagrange equation with $\dot{x}(0) = v$.

The set of vectors v in TM that give rise to static curves $x_v : \mathbb{R} \to M$ is an invariant set that we shall denote by $\hat{\Sigma} := \hat{\Sigma}(L)$. Similarly, the set of vectors v in TM that give rise to semistatic curves $x_v : \mathbb{R} \to M$ is an invariant set that we shall denote by $\Sigma := \Sigma(L)$. As we mentioned before $\hat{\Sigma} \subset \Sigma$. The continuity properties of A_{L+c} and Φ_c imply that Σ and $\hat{\Sigma}$ are closed sets.

Lemma 2.3. *Let $p : M_1 \to M_2$ be a covering such that $c(L_1) = c(L_2)$. Then any lift of a semistatic curve of L_2 is a semistatic curve of L_1. Also the projection of a static curve of L_1 is a static curve of L_2. If in addition, p is a finite covering, then any lift of a static curve of L_2 is a static curve of L_1.*

Proof. Observe first that for any $k \in \mathbb{R}$ we have that

$$\Phi_k^1(x, y) \geqslant \Phi_k^2(px, py),$$

for all x and y in M_1. Hence, if we write $c = c(L_1) = c(L_2)$ we have

$$\Phi_c^1(x, y) \geqslant \Phi_c^2(px, py), \tag{4}$$

for all x and y in M_1.

Suppose now that $x_2 : \mathbb{R} \to M_2$ is a semistatic curve of L_2 and let $x_1 : \mathbb{R} \to M_1$ be any lift of x_2 to M_1. Using (4) and the fact that x_2 is semistatic we have for $s \leqslant t$,

$$\begin{aligned} \Phi_c^1(x_1(s), x_1(t)) &\leqslant A_{L_1+c}(x_1|_{[s,t]}) = A_{L_2+c}(x_2|_{[s,t]}) \\ &= \Phi_c^2(x_2(s), x_2(t)) \leqslant \Phi_c^1(x_1(s), x_1(t)). \end{aligned}$$

Hence, x_1 is semistatic for L_1.

Suppose now that $x_1 : \mathbb{R} \to M_1$ is a static curve of L_1 and let $x_2 : \mathbb{R} \to M_2$ be $p \circ x_1$. Using (4) and the fact that x_1 is static we have for $s \leqslant t$,

$$\begin{aligned} -\Phi_c^1(x_1(t), x_1(s)) &= \Phi_c^1(x_1(s), x_1(t)) = A_{L_1+c}(x_1|_{[s,t]}) = A_{L_2+c}(x_2|_{[s,t]}) \\ &\geqslant \Phi_c^2(x_2(s), x_2(t)) \geqslant -\Phi_c^2(x_2(t), x_2(s)) \geqslant -\Phi_c^1(x_1(t), x_1(s)). \end{aligned}$$

Hence x_2 is static for L_2.

Suppose now that p is a *finite* covering and let $x_2 : \mathbb{R} \to M_2$ be a static curve of L_2. Let $x_1 : \mathbb{R} \to M_1$ be any lift of x_2 to M_1. Since x_2 is static, given $s \leqslant t$ and $\varepsilon > 0$, there exists a curve $\alpha : [0, T] \to M_2$ with $\alpha(0) = x_2(t)$, $\alpha(T) = x_2(s)$ such that

$$A_{L_2+c}(x_2|_{[s,t]}) + A_{L_2+c}(\alpha) \leqslant \varepsilon.$$

Since p is a finite covering, there exists a positive integer n, bounded from above by the number of sheets of the covering, such that the nth iterate of $x_2|_{[s,t]} * \alpha$ lifts to M_1 as a closed curve. Hence, there exists a curve β joining $x_1(t)$ to $x_1(s)$ such that

$$A_{L_1+c}(x_1|_{[s,t]}) + A_{L_1+c}(\beta) \leqslant n\varepsilon,$$

and thus x_1 is static for L_1. $\square$

3. Proof of Theorem A

We shall endow M with a Riemannian metric and we consider in TM the associated Sasaki metric. Let d_M and d_{TM} be the corresponding distance functions of these Riemannian metrics. Given $v \in TM$ denote by $\alpha(v)$ and $\omega(v)$ its α and ω-limits, respectively. We recall the following:

Lemma 3.1 (Contreras et al. [4]). *If $v \in \Sigma$ is semistatic, then $\alpha(v) \subset \hat{\Sigma}$ and $\omega(v) \subset \hat{\Sigma}$. Moreover, $\alpha(v)$ and $\omega(v)$ are each included in a static class.*

Set

$$\Sigma^{\varepsilon} := \{ w \in TM | x_w : [0, \varepsilon) \to M \text{ or } x_w : (-\varepsilon, 0] \to M \text{ is semistatic} \}.$$

Theorem 3.2 (Graph Property, see Mañé [10]; Contreras et al. [4]; Mather [13]). *For all* $p \in \pi(\hat{\Sigma})$ *there exists a unique* $\xi(p) \in T_pM$ *such that* $(p, \xi(p)) \in \Sigma^{\varepsilon}$, *in particular* $(p, \xi(p)) \in \hat{\Sigma}$ *and* $\hat{\Sigma} = graph(\xi)$. *Moreover, there exist positive constants* η *and* K *such that if* $(p, v) \in \hat{\Sigma}$, $(q, w) \in \Sigma^{\varepsilon}$ *and* $d_M(p, q) < \eta$ *then*

$$d_{TM}((p, v),(q, w)) < K\, d_M(p, q).$$

In particular, the map $\xi : \pi(\hat{\Sigma}) \to \Sigma$ *is Lipschitz.*

Using the Graph Property we can define an equivalence relation on $\hat{\Sigma}$ by

$$u, v \in \hat{\Sigma}, \quad u \equiv v \Leftrightarrow d_c(\pi(u), \pi(v)) = 0.$$

The equivalence classes are called *static classes*. Let $\boldsymbol{\Lambda}$ be the set of static classes. Define a reflexive partial order $\preccurlyeq$ in $\boldsymbol{\Lambda}$ by

(a) $\preccurlyeq$ is reflexive.
(b) $\preccurlyeq$ is transitive.
(c) If there is $v \in \Sigma$ with $\alpha(v) \subseteq \Lambda_i$ and $\omega(v) \subseteq \Lambda_j$, then $\Lambda_i \preccurlyeq \Lambda_j$.

Let us begin with the proof of the theorem. We shall prove in Proposition 3.4 below that the static classes are connected. Hence, if we assume that there are only finitely many of them, the connected components of $\hat{\Sigma}$ are finite and must coincide with the static classes. For $\varepsilon > 0$, let $\hat{\Sigma}(\varepsilon)$ be the ε-neighborhood of $\hat{\Sigma}$, i.e.

$$\hat{\Sigma}(\varepsilon) := \{v \in TM \mid d_{TM}(v, \hat{\Sigma}) < \varepsilon\}.$$

Fix $\varepsilon > 0$ small enough such that $\varepsilon < \eta$ where η is the positive constant given by Theorem 3.2 and such that the connected components of $\hat{\Sigma}(\varepsilon)$ are the ε-neighborhoods of the static classes. Thus, for $0 < \delta < \varepsilon$, $\hat{\Sigma}(\delta) = \sum_{i=1}^{N(\varepsilon)} \Lambda_i(\delta)$, where $\Lambda_i(\delta)$ are disjoint open sets containing exactly one static class and the number of components $N(\varepsilon)$ is fixed for all $0 < \delta < \varepsilon$.

Now, suppose that the theorem is false. This means that there exists $\Lambda \in \boldsymbol{\Lambda}$ such that the following two sets are not empty:

$$\mathbb{A} := \bigcup_{\{\Lambda_j \in \boldsymbol{\Lambda} \mid \Lambda \preccurlyeq \Lambda_j\}} \Lambda_j, \quad \mathbb{B} := \bigcup_{\{\Lambda_j \in \boldsymbol{\Lambda} \mid \Lambda \not\preccurlyeq \Lambda_j\}} \Lambda_j.$$

Given $v \in \Sigma$ with $\alpha(v) \subseteq \mathbb{A}$ and $0 < \delta < \varepsilon$, define inductively $s_k(v)$, $t_k(v)$, $T_k(v)$ as follows. Let

$$s_1(v) := \inf\{s \in \mathbb{R} \mid f_s(v) \notin \mathbb{A}(\varepsilon)\} \in \mathbb{R} \cup \{+\infty\}.$$

If $s_k(v) < +\infty$, $k \geqslant 1$, define

$$t_k(v) := \sup\{t < s_k(v) \mid f_t(v) \in \mathbb{A}(\delta)\},$$

$$T_k(v) := \inf\{t > s_k(v) \mid f_t(v) \in \mathbb{A}(\delta)\}.$$

Observe that $s_k(v) < +\infty$ implies that $T_k(v) < +\infty$ because by the definition of $\mathbb{B}$ and the transitivity of $\preccurlyeq$ we have that $\omega(v) \subseteq \mathbb{A}$. Define

$$A_k = A_k(\delta) := \sup\{|T_k(v) - t_k(v)| : v \in \Sigma,\ \alpha(v) \subseteq \mathbb{A},\ s_k(v) < +\infty\},$$

if $s_k(v) = +\infty$ for all $v \in \Sigma$ with $\alpha(v) \subseteq \mathbb{A}$, write $A_\ell(\delta) \equiv 0$ for all $\ell \geqslant k$. Now set

$$s_{k+1}(v) := \inf\{s > T_k(v) \,|\, f_t(v) \notin \mathbb{A}(\varepsilon)\}.$$

Observe that $s_k(v)$, $t_k(v)$ and $T_k(v)$ are invariant under f_t.

We split the rest of the proof of Theorem A into the following claims:

Claim 1. *$A_k(\delta) < +\infty$ for all $k = 1, 2, \ldots$ and all $0 < \delta < \varepsilon$.*

Define

$$\mathbb{M} := \{v \,|\, v \in \Sigma,\ \alpha(v) \subseteq \mathbb{A}\}.$$

Claim 2. (a) $\overline{\mathbb{M}} \cap \mathbb{B} \neq \phi$.
(b) $\limsup_k A_k(\delta) = \sup_k A_k(\delta) = +\infty$.

Claim 3. *There exist sequences $v_n \in \Sigma$, $0 < s_n < t_n$ such that $v_n \to u_1 \in \mathbb{A}$, $f_{s_n}(v_n) \to u_2 \notin \mathbb{A}(\varepsilon)$, $f_{t_n}(v_n) \to u_3 \in \mathbb{A}$* and $d_c(\pi u_1, \pi u_3) = 0$.

We now use Claim 3 to complete the proof of Theorem A. If $u_1 \in \Lambda_j \subseteq \mathbb{A}$, we shall prove that $u_2 \in \Lambda_j \backslash \mathbb{A}(\varepsilon)$, obtaining a contradiction and thus proving Theorem A. It is enough to show that $d_c(\pi u_1, \pi u_2) = 0$. Indeed

$$\begin{aligned} d_c(\pi u_1, \pi u_2) &= \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_1) \\ &\leqslant \Phi_c(\pi u_1, \pi u_2) + \Phi_c(\pi u_2, \pi u_3) + \Phi_c(\pi u_3, \pi u_1) \\ &= \lim_n \left[\Phi_c(\pi v_n, \pi f_{s_n}(v_n)) + \Phi_c(\pi f_{s_n}(v_n), \pi f_{t_n}(v_n))\right] + \Phi_c(\pi u_3, \pi u_1) \\ &= \lim_n \Phi_c(\pi v_n, \pi f_{t_n}(v_n)) + \Phi_c(\pi u_3, \pi u_1) \\ &= d_c(\pi u_1, \pi u_3) = 0, \end{aligned}$$

where the fourth equality holds because v_n is a semistatic vector. □

We need the following

Lemma 3.3 (Contreras et al. [4, Corollary 1.4]). *There exists $A > 0$ such that if $p, q \in M$ and $x \in \mathscr{C}_T(p, q)$ satisfy*
(a) $A_L(x) = \min\{A_L(y) \,|\, y \in \mathscr{C}_T(p, q)\}$;
(b) $A_{L+c}(x) < \Phi_c(p, q) + d_M(p, q)$,
then $|\dot{x}(t)| < A$ for all $t \in [0, T]$.

Proof of Claim 1. Suppose that $A_i < +\infty$ for $i = 1, \ldots, k-1$ and $A_k = +\infty$. The case $k = 1$ is similar. Then there exists $v_n \in \Sigma$, with $\alpha(v_n) \subset \mathbb{A}$ and $T_k(v_n) - t_k(v_n) \to +\infty$. We can assume that $t_k(v_n) = 0$ and that v_n converges (Σ is compact). Let $u = \lim_n v_n \in \partial\mathbb{A}(\delta)$. Then for all n we have

$$m\{t < 0 \,|\, f_t(v_n) \notin \mathbb{A}(\varepsilon)\} \leqslant \sum_{i=1}^{k-1} A_i, \tag{5}$$

where m is the Lebesgue measure on $\mathbb{R}$. We claim that $\alpha(u) \subset \overline{\mathbb{A}}$. To prove the claim it suffices to show that there is a sequence $r_m \to -\infty$ such that $f_{r_m}(u) \in \overline{\mathbb{A}(\varepsilon)}$. (Recall that $\alpha(u)$ must be contained in a unique static class by Lemma 3.1.) Suppose that such a sequence does not exist. This means that there exists $R \leqslant 0$ such that for all $t \leqslant R$, $f_t(u) \notin \overline{\mathbb{A}(\varepsilon)}$. Since $v_n \to u$, there exists n sufficiently large for which $f_t(v_n) \notin \overline{\mathbb{A}(\varepsilon)}$ for all $t \in [R - \sum_{i=1}^{k-1} A_i - 2, R-1]$. This contradicts (5).

Since $f_t(v_n) \notin \mathbb{A}(\varepsilon)$ for $0 < t < T_k(v_n)$ and $T_k(v_n) \to +\infty$, then $f_t(u) \notin \mathbb{A}(\varepsilon)$ for all $t > 0$ and hence $\omega(u) \subseteq \mathbb{B}$. But then the orbit of u contradicts the definition of $\mathbb{B}$. □

Proof of Claim 2. (a) Let $p \in \pi\mathbb{A}$, $q \in \pi\mathbb{B}$. For $n > 0$, let $x_{v_n}:[a_n, b_n] \to M$ be a solution of (E–L) such that $x_{v_n}(a_n) = p$, $x_{v_n}(b_n) = q$ and

$$A_{L+c}(x_{v_n}) \leqslant \Phi_c(p,q) + \frac{1}{n}.$$

This implies that

$$A_{L+c}(x_{v_n}|_{[s,t]}) \leqslant \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \frac{1}{n} \tag{6}$$

for all $a_n \leqslant s \leqslant t \leqslant b_n$. We can assume that

$$\inf\{s > a_n \mid x_{v_n}(s) \in \pi\, \mathbb{B}(\delta)\} = 0,$$

and that the sequence v_n converges (cf. Lemma 3.3). Let $u = \lim_n v_n \in \pi^{-1}(\partial\pi\mathbb{B}(\delta))$. Taking limits in (6), we obtain that $x_u|_{[s,t]}$ is semistatic for all $\liminf_n a_n \leqslant s \leqslant t \leqslant \limsup_n b_n$.

Any limit point w of $\dot{x}_{v_n}(a_n) = f_{a_n}(v_n)$ satisfies $\pi(w) = p \in \pi\mathbb{A}$, and by the Graph Property (Theorem 3.2), $w \in \mathbb{A}$. Similarly, any limit point of $f_{b_n}(v_n)$ is in $\mathbb{B}$. Since $\mathbb{A} \cup \mathbb{B}$ is invariant and $u \notin \mathbb{A} \cup \mathbb{B}$, then $\lim_n a_n = -\infty$, $\lim_n b_n = +\infty$. Hence $u \in \Sigma$. Since $f_t(v_n) \notin \mathbb{B}(\delta)$ for all $a_n \leqslant t < 0$ and $a_n \to -\infty$, then $f_t(u) \notin \mathbb{B}(\delta)$ for all $t < 0$. Hence, $\alpha(u) \subseteq \mathbb{A}$ and thus $u \in \mathbb{M}$. Since $u \in \pi^{-1}(\partial\, \pi\, \mathbb{B}(\delta))$ there exists $z \in \mathbb{B}$ such that $d_M(\pi(u), \pi(z)) \leqslant \delta$. Since $z \in \hat{\Sigma}$ and $u \in \Sigma$ by Theorem 3.2 we have

$$d_{TM}((\pi(u), u), (\pi(z), z)) \leqslant K\,\delta.$$

Thus $u \in \mathbb{M} \cap \mathbb{B}(K\,\delta)$. Letting $\delta \to 0$, we obtain that $\overline{\mathbb{M}} \cap \mathbb{B} \neq \phi$.

(b) By Claim 1 it is enough to show that $\sup_k A_k(\delta) = +\infty$. If $\sup_k A_k(\delta) < T$, then $\mathbb{M} \subseteq \mathbb{M}(\delta, T)$, where $\mathbb{M}(\delta, T)$ is the compact set given by

$$\mathbb{M}(\delta, T) = \{v \in \Sigma \,|\, f_{[-T,T]}(v) \cap \overline{\mathbb{A}(\delta)} \neq \phi\} = f_{[-T,T]}(\overline{\mathbb{A}(\delta)} \cap \Sigma).$$

Note that $\mathbb{M}(\delta, T) \cap \mathbb{B} = \phi$, because $\mathbb{B}$ is invariant and $\mathbb{B} \cap \mathbb{A}(\delta) = \phi$. On the other hand, since $\mathbb{M} \subseteq \mathbb{M}(\delta, T)$ we have $\overline{\mathbb{M}} \cap \mathbb{B} \subseteq \mathbb{M}(\delta, T) \cap \mathbb{B} = \phi$. This contradicts item (a). □

Proof of Claim 3. Given $0 < \delta < \varepsilon$, by Claim 2(b) there exists $k > N(\varepsilon)$ such that $A_k(\delta) > 0$. Hence, there is $v = v_\delta \in \Sigma$ with $\alpha(v) \subset \mathbb{A}$, such that the orbit of v leaves $\mathbb{A}(\varepsilon)$ and returns to $\mathbb{A}(\delta)$ at least k times. Since $k > N(\varepsilon)$ there is one component $\Lambda_j(\delta) \subseteq \mathbb{A}(\delta)$ with two of these returns, i.e. there exist $\tau_1(\delta) < s(\delta) < \tau_2(\delta)$ with $f_{\tau_1}(v) \in \Lambda_j(\delta)$, $f_s(v) \notin \mathbb{A}(\varepsilon)$ and $f_{\tau_2}(v) \in \Lambda_j(\delta)$. We can choose v_δ so that $\tau_1(\delta) = 0$. Now, there exists a sequence $\delta_n \downarrow 0$ such that the repeated component $\Lambda_j \subset \Lambda_j(\delta_n)$ is always the same. Let $s_n := s(\delta_n)$, $t_n := \tau_2(\delta_n)$, $v_n := v_{\delta_n}$ and choose a subsequence such that $v_n, f_{s_n}(v_n)$ and $f_{t_n}(v_n)$ converge. Let $u_1 = \lim_n v_n \in \cap_n \Lambda_j(\delta_n) = \Lambda_j$, $u_3 = \lim_n f_{t_n}(v_n) \in \Lambda_j$ and $u_2 = \lim_n f_{s_n}(v_n) \notin \mathbb{A}(\varepsilon)$. Since $u_1, u_3 \in \Lambda_j$, then $d_c(\pi u_1, \pi u_3) = 0$. $\square$

Proposition 3.4. *Every static class is connected.*

Proof. Let Λ be a static class and suppose that it is not connected. Let U_1, U_2 be disjoint open sets such that $\Lambda \subseteq U_1 \cup U_2$ and $\Lambda \cap U_i \neq \phi$, $i = 1,2$. Let $p_i \in \pi(U_i \cap \Lambda)$, $i = 1,2$. Since U_1 and U_2 are disjoint sets we can take a solution $x_{v_n} : [a_n, b_n] \to M$, $a_n < 0 < b_n$ of (E–L) such that $x_{v_n}(0) \notin \pi(U_1 \cup U_2)$, $x_{v_n}(a_n) = p_1$, $x_{v_n}(b_n) = p_2$ and

$$A_{L+c}(x_{v_n}) \leqslant \Phi_c(p_1, p_2) + \frac{1}{n}. \tag{7}$$

Let u be a limit point of v_n, then $x_u : \mathbb{R} \to M$ is semistatic (see the proof of Claim 2 item (a)). Then, for $a_n \leqslant s \leqslant t \leqslant b_n$,

$$d_c(p_1, p_2) \leqslant \Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2) + \Phi_c(p_2, p_1),$$

therefore

$$\begin{aligned} d_c(p_1, p_2) &\leqslant \liminf_n \, [\Phi_c(p_1, x_{v_n}(s)) + \Phi_c(x_{v_n}(s), x_{v_n}(t)) + \Phi_c(x_{v_n}(t), p_2)] + \Phi_c(p_2, p_1) \\ &\leqslant \liminf_n \, A_{L+c}(x_{v_n}) + \Phi_c(p_2, p_1) \\ &\leqslant d_c(p_1, p_2) = 0, \end{aligned}$$

where in the last inequality we used (7). Hence,

$$\Phi_c(p_1, x_u(s)) + \Phi_c(x_u(s), x_u(t)) + \Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) = 0.$$

Combining the last equation with the triangle inequality we obtain

$$d_c(x_u(s), x_u(t)) \leqslant \Phi_c(x_u(s), x_u(t)) + [\Phi_c(x_u(t), p_2) + \Phi_c(p_2, p_1) + \Phi_c(p_1, x_u(s))] = 0.$$

So that $u \in \hat{\Sigma}$. Moreover, for $s = 0, t = 1$:

$$d_c(x_u(0), p_1) \leqslant \Phi_c(p_1, x_u(0)) + [\Phi_c(x_u(0), x_u(1)) + \Phi_c(x_u(1), p_2) + \Phi_c(p_2, p_1)] = 0.$$

Hence $x_u(0) \in \pi(\Lambda)$. On the other hand, $x_u(0) \notin \pi(U_1 \cup U_2)$. This contradicts the fact that $\Lambda \subseteq U_1 \cup U_2$. $\square$

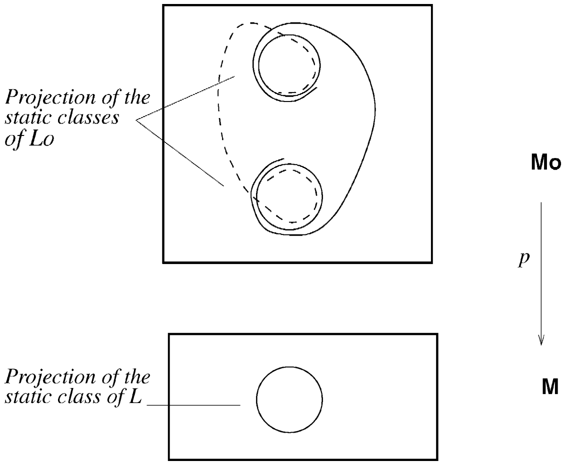

Fig. 2. Creating homoclinic connections with finite coverings and Theorem A.

4. Proof of Theorem B

Let $U \stackrel{\text{def}}{=} U_\varepsilon$ denote the ε-neighborhood of $\pi(\hat{\Sigma}(L))$, where $\hat{\Sigma}(L)$ is the set of static vectors of L. Since we are assuming that $\hat{\Sigma}(L)$ contains only one static class, the set U is also connected for small ε. Let $i: U \to M$ be the inclusion map. The vector space $H_1(M, U, \mathbb{R})$ is isomorphic to the quotient of $H_1(M,\mathbb{R})$ by $i_*(H_1(U,\mathbb{R}))$.

Let G be the quotient of $H_1(M, U, \mathbb{Z})$ by its torsion part. Since G is free we can write $G = \mathbb{Z} \oplus \overset{k}{\ldots} \oplus \mathbb{Z}$, where $k = \dim H_1(M, U, \mathbb{R})$. The group G can be seen as a lattice in $H_1(M, U, \mathbb{R})$. Let J be a finite index subgroup of G. There is a surjective homomorphism $j: G \to G/J$ given by the projection.

If we take the Hurewicz map

$$\pi_1(M) \mapsto H_1(M, \mathbb{Z}),$$

and we compose it with the surjective homomorphisms $H_1(M, \mathbb{Z}) \mapsto H_1(M, U, \mathbb{Z})$, $H_1(M, U, \mathbb{Z}) \mapsto G$ and $j: G \to G/J$, we obtain a surjective homomorphism

$$\pi_1(M) \mapsto G/J,$$

whose kernel will be the fundamental group of a finite Galois covering M_0 of M with covering projection map $p: M_0 \to M$ and group of deck transformations given by the finite abelian group G/J.

Observe that G/J acts transitively and freely on the set of connected components of $p^{-1}(U)$ which coincides with the set of connected components of $p^{-1}(\pi(\hat{\Sigma}(L)))$. Therefore, we have

Lemma 4.1. *There is a one to one correspondence between elements in G/J and connected components of $p^{-1}(\pi(\hat{\Sigma}(L)))$.*

Observe that to each homoclinic orbit $x:\mathbb{R}\to M$ to $\hat{\Sigma}(L)$ we can associate a homology class in G. Indeed, since there exists $t_0>0$ such that for all t with $|t|\geqslant t_0$, $x(t)\in U$, the class of $x|_{[-t_0,t_0]}$ defines an element in $H_1(M,U,\mathbb{Z})$ and hence in G. Let us denote by $\mathscr{H}$ the subset of G given by all the classes corresponding to homoclinic orbits to $\hat{\Sigma}(L)$.

Lemma 4.2. *For any J as above, the image of $\langle\mathscr{H}\rangle$ under j is precisely G/J.*

Proof. Let L_0 denote the lift of the Lagrangian L to M_0. Observe first that by Lemma 2.2, $c(L)=c(L_0)$ and therefore by Lemma 2.3 we have

$$\pi_0(\hat{\Sigma}(L_0))=p^{-1}(\pi(\hat{\Sigma}(L))), \tag{8}$$

where $\pi_0:TM_0\to M_0$ is the canonical projection of the tangent bundle TM_0 to M_0.

Let us prove now that L_0 satisfies the hypothesis of Theorem A, that is, the number of static classes of L_0 is finite. In fact, we shall show that the projection to M_0 of a static class of L_0 coincides with a connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$. Using (8) and Proposition 3.4 we see that the projection of a static class of L_0 to M_0 must be contained in a single connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$. Hence, it suffices to show that if x and y belong to a connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$ then $d_c^0(x,y)=0$. Since we are assuming that $\hat{\Sigma}(L)$ contains only one static class we have that $d_c(px,py)=0$. Since $p:M_0\to M$ is a finite covering there are lifts x_1 of px and y_1 of py such that $d_c^0(x_1,y_1)=0$. Since static classes are connected x_1 and y_1 must belong to the same connected component of $p^{-1}(\pi(\hat{\Sigma}(L)))$ and thus there is a covering transformation taking x_1 into x and y_1 into y which implies that $d_c^0(x,y)=0$ as desired.

Now Theorem A and (8) imply that every covering transformation in G/J can be written as the composition of covering transformations that arise from elements in $\mathscr{H}$, that is, $j(\langle\mathscr{H}\rangle)=G/J$. $\square$

We shall need the following algebraic lemma.

Lemma 4.3. *Let $G=\mathbb{Z}\oplus\overset{k}{\cdots}\oplus\mathbb{Z}$. Given a finite index subgroup $J\subset G$ let us denote by $j:G\to G/J$ the projection homomorphism.*

Let A be a subgroup of G. If A has the property that for all J as above $j(A)=G/J$, then $A=G$.

Proof. The hypothesis readily implies that

$$A/A\cap J \text{ is isomorphic to } G/J. \tag{9}$$

- If the rank of A is strictly less than the rank of G, one can easily construct a subgroup $J\subset G$ with finite index such that $A\subseteq J$ and $G/J\neq\{0\}$. But this contradicts (9) because $A/A\cap J=\{0\}$.

- If the rank of A equals the rank of G, then A has finite index in G and by (9) $G/A = \{0\}$ and thus $G = A$. $\square$

Observe now that any set $\mathscr{H}$ of a free abelian group G of rank k such that the group generated by $\mathscr{H}$ is G must have at least k elements. Therefore, if we combine Lemmas 4.2 and 4.3 with $\langle \mathscr{H} \rangle = A$, we deduce that the set $\mathscr{H}$ of classes corresponding to homoclinic orbits generates G and must have at least k elements thus concluding the proof of Theorem B. $\square$

5. Proof of Theorem C

To prove the theorem we shall show first several lemmas. We will use the following notation:

- $\mathscr{M}^0(L) =$ minimizing measures of L;
- $\Sigma(L) =$ semistatic vectors of L;
- $\hat{\Sigma}(L) =$ static vectors of L;
- $\Lambda(L) =$ closure of $\bigcup_{\mu \in \mathscr{M}^0(L)} \mathrm{supp}(\mu)$.

Recall that we always have $\Lambda(L) \subseteq \hat{\Sigma}(L) \subseteq \Sigma(L)$.

Lemma 5.1. *The function* $C^\infty(M,\mathbb{R}) \ni \psi \mapsto c(L+\psi)$ *is continuous.*

Proof. Suppose that $\psi_n \to \psi$ and let $c_n := c(L+\psi_n)$ and $c := c(L+\psi)$. We will prove that $c_n \to c$.
Fix $\varepsilon > 0$. Since $c - \varepsilon < c$, by the definition of critical value there exists a closed curve $\gamma:[0,T] \to M$ such that

$$A_{L+\psi+c-\varepsilon}(\gamma) < 0,$$

hence for all n sufficiently large

$$A_{L+\psi_n+c-\varepsilon}(\gamma) < 0,$$

therefore for all n sufficiently large

$$c - \varepsilon < c_n,$$

and thus

$$c - \varepsilon \leqslant \liminf_n c_n.$$

Since ε was arbitrary we have

$$c \leqslant \liminf_n c_n.$$

Let us show now that $\limsup_n c_n \leqslant c$. Suppose that $c < \limsup_n c_n$. Take ε such that

$$c < c + \varepsilon < \limsup_n c_n. \tag{10}$$

Since $\psi_n \to \psi$, there exists n_0 such that for all $n \geqslant n_0$,

$$-\varepsilon \leqslant \psi_n - \psi \leqslant \varepsilon. \tag{11}$$

By (10), there exists $m \geqslant n_0$ such that

$$c < c + \varepsilon < c_m.$$

By the definition of critical value there exists a closed curve $\gamma : [0, T] \to M$ such that

$$A_{L+\psi_m+c+\varepsilon}(\gamma) < 0,$$

and hence using (11) we have

$$A_{L+\psi+c}(\gamma) \leqslant A_{L+\psi_m+c+\varepsilon}(\gamma) < 0,$$

which yields a contradiction to the definition of the critical value c. □

This proof also shows that $L \mapsto c(L)$ is continuous if we endow the set of Lagrangians L with the topology of uniform convergence on compact subsets of TM.

Lemma 5.2. $\lim_{\psi\to 0} \Sigma(L + \psi) \subset \Sigma(L)$, *where* $\lim_{\psi\to 0} \Sigma(L + \psi)$ *is the set of accumulation points of sequences* $v_n \in \Sigma(L + \psi_n) \subset TM$ *with* $\psi_n \to 0$.

Proof. Let $\psi_n \to 0$ and $v_n \in \Sigma(L + \psi_n)$ with $v_n \to v$. Let $T > 0$ and write $x_{v_n}(t) := \pi \circ f_t^n(v_n)$, $x_v(t) := \pi \circ f_t(v)$, $x_n = x_{v_n}(-T)$, $y_n = y_{v_n}(T)$, $x = x_v(-T)$, $y = x_v(T)$, $c_n = c(L + \psi_n)$ and $c = c(L)$, where f_t^n and f_t are the Euler–Lagrange flows of $L + \psi_n$ and L, respectively. Then

$$\Phi_c(x, y) \leqslant A_{L+c}(x_v|_{[-T,T]}) = \lim_n A_{L+\psi_n+c_n}(x_{v_n}|_{[-T,T]}) = \lim_n \Phi^n_{c_n}(x_n, y_n), \tag{12}$$

where Φ^n and Φ are the action potentials of $L + \psi_n$ and of L, respectively. Write $\Delta := \lim_n \Phi^n_{c_n}(x_n, y_n)$. We shall prove that $\Delta = \Phi_c(x, y)$, then (12) becomes an equality and hence $x_v|_{[-T,T]}$ is semistatic. Since $T > 0$ is arbitrary, then $v \in \Sigma(L)$.

Suppose that $\Phi_c(x, y) < \Delta - \varepsilon$, then there exists a curve $\eta : [0, S] \to M$ with $\eta(0) = x$, $\eta(S) = y$ and $A_{L+c}(\eta) < \Delta - \varepsilon$. Then

$$\Phi^n_{c_n}(x_n, y_n) \leqslant A_{L+\psi_n+c_n}(\eta) + \Phi^n_{c_n}(x, x_n) + \Phi^n_{c_n}(y, y_n). \tag{13}$$

Fix a Riemannian metric on M. Using a speed 1 geodesic from $z_1 \in M$ to $z_2 \in M$, we get that

$$\Phi^n_{c_n}(z_1, z_2) \leqslant \left(\max_{(x,v)\in TM:\ |v|=1} |L(x, v)| + \max_{x\in M} |\psi_n(x)| + c_n \right) d_M(z_1, z_2).$$

Hence, there exist $K > 0$ such that for all n sufficiently large we have $\Phi^n_{c_n}(z_1, z_2) \leqslant K\ d_M(z_1, z_2)$. Letting $n \to \infty$ on equation (13), we get that $\lim_n \Phi^n_{c_n}(x_n, y_n) \leqslant \Delta - \varepsilon$. This contradicts the definition of Δ. □

Lemma 5.3. *If $\mathscr{M}^0(L) = \{\mu\}$ then $\hat{\Sigma}(L) = \Sigma(L)$.*

Proof. We show first that $\mathrm{supp}(\mu)$ is inside a static class. Since $\mathscr{M}^0(L) = \{\mu\}$, then μ is ergodic. In particular, μ-almost every point has a dense orbit on $\mathrm{supp}(\mu)$. Let $v \in \mathrm{supp}(\mu)$ be such that it has a dense orbit on $\mathrm{supp}(\mu)$. Let $u, w \in \mathrm{supp}(\mu)$ and let $0 < r_n < s_n < t_n$ be such that $\lim_n f_{r_n}(v) = u = \lim_n f_{t_n}(v)$ and $\lim_n f_{s_n}(v) = w$. Then

$$\begin{aligned} d_c(\pi f_{r_n}(v), \pi f_{t_n}(v)) &\leqslant A_{L+c}(\pi f_{[r_n,s_n]}(v)) + A_{L+c}(\pi f_{[s_n,t_n]}(v)) \\ &= A_{L+c}(\pi f_{r_n}(v), \pi f_{t_n}(v)) \\ &= \Phi_c(\pi f_{r_n}(v), \pi f_{t_n}(v)) \end{aligned}$$

and

$$\begin{aligned} d_c(\pi u, \pi w) &= \lim_n d_c(\pi f_{r_n}(v), \pi f_{t_n}(v)) \\ &\leqslant \lim_n \Phi_c(\pi f_{r_n}(v), \pi f_{t_n}(v)) = \Phi_c(\pi u, \pi u) = 0, \end{aligned}$$

and hence $\mathrm{supp}(\mu)$ is inside a static class.

Now let $v \in \Sigma(L)$. For $S, T > 0$ consider the probabilities v_{ST} defined by

$$\int g \, \mathrm{d}v_{ST} = \frac{1}{S+T} \int_{-S}^{T} g(f_s(v)) \, \mathrm{d}s,$$

for any $g: TM \to \mathbb{R}$ continuous. Since the ω and α-limits of v are in $\hat{\Sigma}(L)$ and any weak limit of $\{v_{ST}\}_{S,T>0}$ is invariant, then by Theorem IV in [10,4], any weak* limit of v_{ST} is minimizing and hence it is μ.

Then $\alpha\text{-limit}(v) \subset \mathrm{supp}(\mu)$ and $\omega\text{-limit}(v) \subset \mathrm{supp}(\mu)$. Let $u \in \alpha\text{-limit}(v)$, $w \in \omega\text{-limit}(v)$ and $S_n, T_n \to +\infty$ such that $\lim_n f_{-S_n}(v) = u$ and $\lim_n f_{T_n}(v) = w$. For $s, t > 0$ define

$$\delta(s,t) = A_{L+c}(\pi f_{[-s,t]}(v)) + \Phi_c(\pi f_t(v), f_{-s}(v)). \tag{14}$$

Then the triangle inequality for Φ_c implies that $\delta(-s,t)$ is increasing on $s > 0$ and $t > 0$. Also, since v is semistatic, $\delta(-s,t) = d_c(\pi f_{-s}(v), \pi f_t(v)) \geqslant 0$. But then, since $\mathrm{supp}(\mu)$ is inside a static class,

$$\lim_n \delta(-S_n, T_n) = d_c(\pi u, \pi w) = 0.$$

Hence, $\delta(-s,t) \equiv 0$ for all $s, t > 0$, and thus Eq. (14) implies that $v \in \hat{\Sigma}(L)$. □

Lemma 5.4. *Let*

$$\mathscr{G}_2 := \{\psi \in C^\infty(M,\mathbb{R}) \,|\, \mathscr{M}^0(L+\psi) = \{\mu\} \text{ and } \hat{\Sigma}(L+\psi) = \mathrm{supp}(\mu)\}.$$

Then

(a) *$\mathscr{G}_2$ is dense in $C^\infty(M,\mathbb{R})$.*
(b) *If $\psi_0 \in \mathscr{G}_2$, then* $\lim_{\psi \to \psi_0} d_{\mathrm{H}}(\hat{\Sigma}(L+\psi), \hat{\Sigma}(L+\psi_0)) = 0$ *where d_{H} is the Hausdorff metric between compact subsets of TM.*

(c) *If* $\psi \in C^\infty(M,\mathbb{R})$, $\mu_\psi \in \mathscr{M}^0(L+\psi)$ *and* $\psi_0 \in \mathscr{G}_2$, *then*

$$\lim_{\psi\to\psi_0} d_{\mathrm{H}}(\mathrm{supp}(\mu_\psi),\, \mathrm{supp}(\mu_{\psi_0})) = 0.$$

Proof. Let us prove (a). By Theorem C in [11], the set

$$\mathscr{G}_1 := \{\psi \in C^\infty(M,\mathbb{R}) \mid \# \mathscr{M}^0(L+\psi) = 1\}$$

is generic in $C^\infty(M,\mathbb{R})$. We shall see that if $\psi_0 \in \mathscr{G}_1$, $\mathscr{M}^0(L+\psi_0) = \{\mu\}$ and $\psi_1 \in C^\infty(M,\mathbb{R})$ is such that $\psi_1 \geqslant 0$ and $\{x : \psi_1(x) = 0\} = \pi(\mathrm{supp}(\mu))$, then $\hat{\Sigma}(L+\psi_0+\psi_1) = \mathrm{supp}(\mu)$ and $\mathscr{M}^0(L+\psi_0+\psi_1) = \{\mu\}$. This implies that $\mathscr{G}_2$ is dense in $C^\infty(M,\mathbb{R})$.

Observe that $\mu \in \mathscr{M}^0(L+\psi_0+\psi_1)$ and hence $\mathrm{supp}(\mu) \subseteq \hat{\Sigma}(L+\psi_0+\psi_1)$. If $v_0 \in TM$ and $\pi(v_0) \notin \pi(\mathrm{supp}(\mu))$, we shall see that $v_0 \notin \hat{\Sigma}(L+\psi_0+\psi_1)$. Then the Graph Property (cf. Theorem 3.2) implies that $\hat{\Sigma}(L+\psi_0+\psi_1) = \mathrm{supp}(\mu)$. Indeed, if $v_1 = f_1^{L+\psi_0+\psi_1}(v_0)$, then

$$\begin{aligned}
&A_{L+\psi_0+\psi_1+c}(\pi f_{[0,1]}^{L+\psi_0+\psi_1}(v_0)) + \Phi_c^{L+\psi_0+\psi_1}(\pi v_1, \pi v_0)\\
&\quad\geqslant \int_0^1 \psi_1(\pi f_s^{L+\psi_0+\psi_1}(v_0))\,\mathrm{d}s + \Phi_c^{L+\psi_0}(\pi v_0, \pi v_1) + \Phi_c^{L+\psi_0}(\pi v_1, \pi v_0)\\
&\quad> d_c^{L+\psi_0}(\pi v_0, \pi v_1) \geqslant 0.
\end{aligned}$$

Hence v_0 is not static.

We now prove (b) and (c). From Lemmas 5.2 and 5.3, if $\psi_0 \in \mathscr{G}_2$, then $\lim_{\psi\to\psi_0} \hat{\Sigma}(L+\psi) \subseteq \hat{\Sigma}(L+\psi_0) = \mathrm{supp}(\mu_{\psi_0})$. On the other hand the continuity of the critical value ensures that the weak* limit of minimizing measures of $L+\psi$ is minimizing for $L+\psi_0$ and hence $\lim_{\psi\to\psi_0} \hat{\Sigma}(L+\psi) \supseteq \hat{\Sigma}(L+\psi_0)$ and thus $\lim_{\psi\to\psi_0} \hat{\Sigma}(L+\psi) = \mathrm{supp}(\mu_{\psi_0})$.

This implies that for any neighborhood U of $\mathrm{supp}(\mu_{\psi_0})$ there is a neighborhood $\mathscr{V}$ of ψ_0 such that $\hat{\Sigma}(L+\psi) \subseteq U$ for all $\psi \in \mathscr{V}$. Let d be the distance function of some Riemannian metric on TM. Using neighborhoods

$$U_\varepsilon := \{z \in TM \mid d(z,\, \mathrm{supp}(\mu_{\psi_0})) < \varepsilon\},$$

one gets that

$$\lim_{\psi\to\psi_0} \sup_{z\in\Lambda(L+\psi)} d(z,\, \mathrm{supp}(\mu_{\psi_0})) \leqslant \lim_{\psi\to\psi_0} \sup_{z\in\hat{\Sigma}(L+\psi)} d(z,\, \Sigma(L+\psi_0)) = 0. \tag{15}$$

Given $\varepsilon > 0$, let $\{z_1,\ldots,z_N\} \subset \mathrm{supp}(\mu_{\psi_0})$ be such that $\mathrm{supp}(\mu_{\psi_0}) \subset \bigcup_{i=1}^N B(z,\varepsilon_i)$, where $B(z,\varepsilon) := \{w \in TM \mid d(z,w) < \varepsilon\}$ and let $g_i : TM \to [0,1]$ be a non-constant positive continuous function with $\mathrm{supp}(g_i) \subseteq B(z_i,\varepsilon)$. Then $\int g_i\,\mathrm{d}\mu_{\psi_0} > 0$. The continuity of $c(L)$ implies that if $\psi \to \psi_0$ and $\mu_\psi \in \mathscr{M}^0(L+\psi)$ then $\mu_\psi \to \mu_{\psi_0}$ weakly*. Hence, there is a neighborhood $\mathscr{V}$ of ψ_0 such that if $\psi \in \mathscr{V}$ and $\mu_\psi \in \mathscr{M}^0(L+\psi)$, then $\int g_i\, d\mu_\psi > 0$ for all $i = 1,\ldots,N$. Hence,

$$\lim_{\psi\to\psi_0} \sup_{z\in\mathrm{supp}(\mu_{\psi_0})} d(z,\, \Lambda(L+\psi)) \leqslant \varepsilon.$$

Since this holds for any $\varepsilon > 0$, then

$$\lim_{\psi \to \psi_0} \sup_{z \in \hat{\Sigma}(L+\psi_0)} d(z, \hat{\Sigma}(L+\psi)) \leqslant \lim_{\psi \to \psi_0} \sup_{z \in \mathrm{supp}(\mu_{\psi_0})} d(z, \Lambda(L+\psi)) = 0. \tag{16}$$

From (15) and (16) we get that

$$\lim_{\psi \to \psi_0} d_{\mathrm{H}}(\hat{\Sigma}(L+\psi), \hat{\Sigma}(L+\psi_0)) = \lim_{\psi \to \psi_0} d_{\mathrm{H}}(\Lambda(L+\psi), \mathrm{supp}(\mu_{\psi_0})) = 0. \qquad \square$$

To complete the proof of Theorem C we now show that $\mathscr{G}_2$ is generic.

We claim that the set

$$\mathscr{U}(\varepsilon) := \{\psi \in C^\infty(M, \mathbb{R}) \,|\, d_{\mathrm{H}}(\hat{\Sigma}(L+\psi), \Lambda(L+\psi)) < \varepsilon\}$$

contains a neighborhood of $\mathscr{G}_2$.

This follows from parts (b) and (c) in Lemma 5.4 and the triangle inequality for the Hausdorff distance, i.e. using that $\hat{\Sigma}(L+\psi_0) = \mathrm{supp}(\mu_{\psi_0})$ for $\psi_0 \in \mathscr{G}_2$, we have that

$$\begin{aligned} &d_{\mathrm{H}}(\hat{\Sigma}(L+\psi), \Lambda(L+\psi)) \\ &\quad \leqslant d_{\mathrm{H}}(\hat{\Sigma}(L+\psi), \hat{\Sigma}(L+\psi_0)) + d_{\mathrm{H}}(\mathrm{supp}(\mu_{\psi_0}), \Lambda(L+\psi)). \end{aligned}$$

Since $\mathscr{G}_2$ is dense, the set $\mathscr{U}(\varepsilon)$ contains a open and dense set. Then

$$\bigcap_{n>0} \mathscr{U}\left(\frac{1}{n}\right) = \{\psi \in C^\infty(M, \mathbb{R}) \,|\, \hat{\Sigma}(L+\psi) = \Lambda(L+\psi)\}$$

is generic. Since $\mathscr{G}_2 = \mathscr{G}_1 \cap [\bigcap_{n>0} \mathscr{U}(1/n)]$ and $\mathscr{G}_1$ is generic, then $\mathscr{G}_2$ is generic. $\square$

6. Proof of Corollaries 2 and 3

We need the following easy lemma.

Lemma 6.1. *Let M be a closed manifold with first Betti number $b_1(M,\mathbb{R}) \geqslant 2$. Then if $A \subset M$ is a closed submanifold diffeomorphic to S^1 and U_ε denotes the ε neighborhood of A, we have that $H_1(M, U_\varepsilon, \mathbb{R})$ is non zero for all ε sufficiently small.*

Proof. Since A is diffeomorphic to a circle, the singular homology of the pair (M, U_ε) coincides with the singular homology of the pair (M, A) and therefore the vector space $H_1(M, U_\varepsilon, \mathbb{R})$ must have dimension $\geqslant b_1(M, \mathbb{R}) - 1 \geqslant 1$. $\square$

We recall the following generic property proved in [5,11] that we already mentioned in the introduction.

Theorem 6.2. *Given a Lagrangian L there exists a generic set $\mathcal{O} \subset C^\infty(M,\mathbb{R})$ such that if $\psi \in \mathcal{O}$ the Lagrangian $L + \psi$ has a unique minimizing measure in $\mathscr{M}^0(L+\psi)$ and this measure is uniquely ergodic. When this measure is supported on a periodic orbit, this orbit is hyperbolic and if the stable and unstable manifolds intersect, they must do it transversally.*

It is conjectured in [10] that the unique minimizing measure in $\mathcal{M}^0(L+\psi)$ is always supported on a periodic orbit.

Observe now that if we combine Corollary 1, Lemma 6.1 and Theorem 6.2 we obtain Corollary 2.

To prove Corollary 3 we need the following lemma. A proof can be found in [8, Proposition 8].

Lemma 6.3. *If L is a symmetric Lagrangian, then*

$$c(L) = -\inf_{x\in M} L(x,0),$$

and

$$\Lambda(L) = \hat{\Sigma}(L) = \{(x,0)\colon L(x,0) = -c(L)\}.$$

Moreover, the ergodic minimizing measures are the Dirac measures concentrated on the fixed points $(x,0)$ *of the Euler–Lagrange flow with* $L(x,0) = -c(L)$.

Finally observe that if we combine Corollary 1, Lemma 6.3 and Theorem 6.2 we obtain Corollary 3. □

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References

[1] S. Bolotin, Invariant sets of Hamiltonian systems and variational methods, Proceedings of the International Congress of Mathematicians, Vol. 1,2, Zürich, 1994, Birkhäuser, Basel, 1995, pp. 1169–1178.
[2] S. Bolotin, Homoclinic orbits in invariant tori of Hamiltonian systems, in: V.V. Kozlov (Ed.), Dynamical Systems in Classical Mechanics, American Mathematical Society Translation Series 2, Vol. 168, American Mathematics Society, Providence, RI, 1995, pp. 21–90.
[3] S. Bolotin, Homoclinic trajectories of invariant sets of Hamiltonian systems, Nonlinear Differential Equation Appl. 4 (3) (1997) 359–389.
[4] G. Contreras, J. Delgado, R. Iturriaga, Lagrangian flows: the dynamics of globally minimizing orbits II, Bol. Soc. Bras. Mat. 28 (2) (1997) 155–196.
[5] G. Contreras, R. Iturriaga, Convex Hamiltonians without conjugate points, Ergodic Theory Dynamical System 19 (1999) 901–952.
[6] A. Fathi, Théorème KAM faible et Théorie de Mather sur les systems Lagrangiens, C.R. Acad. Sci. Paris, Sér. I 324 (1997) 1043–1046.
[7] A. Fathi, Solutions KAM faibles conjuguées et barrières de Peierls, C.R. Acad. Sci. Paris, Sér. I 325 (1997) 649–652.
[8] A. Fathi, Orbites hétéroclines et ensemble de Peierls, C. R. Acad. Sci. Paris, Sér. I 326 (1998) 1213–1216.
[9] A. Fathi, Weak KAM theorem in Lagrangian dynamics, Lecture Notes, ENS Lyon, 2000.

[10] R. Mañé, Lagrangian flows: the dynamics of globally minimizing orbits, in: F. Ledrappier, J. Lewowicz, S. Newhouse (Eds.), International Congress on Dynamical Systems in Montevideo (a tribute to Ricardo Mañé), Pitman Research Notes in Mathematics, Vol. 362, 1996, pp. 120–131. (reprinted in Bol. Soc. Bras. Mat. 28 (2) (1997) 141–153.
[11] R. Mañé, Generic properties and problems of minimizing measures of Lagrangian systems, Nonlinearity 9 (1996) 273–310.
[12] J. Mather, Action minimizing measures for positive definite Lagrangian systems, Math. Z. 207 (1991) 169–207.
[13] J. Mather, Variational construction of connecting orbits, Ann. Inst. Fourier 43 (1993) 1349–1386.